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2006 J. Phys. A: Math. Gen. 39 4945

(http://iopscience.iop.org/0305-4470/39/18/012)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 4945–4953 doi:10.1088/0305-4470/39/18/012

On the discrete Frobenius–Perron operator of the Bernoulli map

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Received 30 November 2005, in final form 15 March 2006
Published 19 April 2006
Online at stacks.iop.org/JPhysA/39/4945

Abstract
We study the spectra of a finite-dimensional Frobenius–Perron operator (matrix) of the Bernoulli map derived from phase space discretization. The eigenvalues and (right and left) eigenvectors are analytically calculated, which are closely related to periodic orbits on the partition points. In the degenerate case, Jordan decomposition of the matrix is explicitly constructed. Except for the isolated eigenvalue 1, there is no definite limit with respect to eigenvalues when $n \to \infty$. The behaviour of the eigenvectors is discussed in the limit of large n.

PACS numbers: 02.10.Ud, 02.50.−r, 05.45.Ac

1. Introduction

The Bernoulli map, $F(x) = 2x \pmod 1$, is a basic pedagogical model of chaotic dynamics. In this simple system, the exponential sensitivity to the initial condition is obvious and the exponential proliferation of periodic orbit (PO) can be easily identified. When turning to statistical description of dynamics, however, the situation is far from obvious. The evolution of probability densities is described by the Frobenius–Perron (F–P) operator, which transforms a distribution ρ to $\hat{U}\rho$ with

$$(\hat{U}\rho)(x) = \int_0^1 \delta(x - F(x'))\rho(x')\,\mathrm{d}x' = \frac{1}{2}\left[\rho\left(\frac{x}{2}\right) + \rho\left(\frac{1+x}{2}\right)\right]. \tag{1.1}$$

The equilibrium state $\rho_0(x) = 1$ is an eigenvector of $\hat{U}$ with eigenvalue $\lambda_0 = 1$, while the eigenvalues within the unit circle determine how fast the equilibrium state is attained. As an important property of F–P operators, the spectra of $\hat{U}$ are crucially controlled by the function space on which it acts [1, 2]. For example, if the function space is $L^2[0, 1]$, then any z within the unit circle is an eigenvalue with infinite multiplicity, which indicates that the approaching to equilibrium can be arbitrarily slow. In contrast, if $\hat{U}$ is restricted to the space of analytic functions, there are only countable non-zero eigenvalues $\lambda_n = 2^{-n}$, $n = 0, 1, 2, \ldots$, and the eigenvector corresponding to λ_n, denoted by $B_n(x)$, is known as the Bernoulli polynomial,

$$B_0(x) = 1 \qquad B_1(x) = x - \tfrac{1}{2} \qquad B_2(x) = x^2 - x + \tfrac{1}{6} \cdots. \tag{1.2}$$

Moreover, the corresponding left eigenvectors in this case are generalized functions, i.e. δ-function and its derivatives. From the discrete eigenvalues and (right and left) eigenvectors, a generalized spectral decomposition of $\hat{U}$ can be constructed, by which the exponential decorrelation of polynomial observables can be conveniently calculated [3–5].

In order to achieve a better understanding of the above intriguing facts, in this paper we consider the finite-dimensional approximation of $\hat{U}$. The reduction is based on a straightforward discretization of phase space, i.e.

$$[0, 1] = \bigcup_{j=0}^{n-1} \left[\frac{j}{n}, \frac{j+1}{n}\right] \equiv \bigcup_{j=0}^{n-1} I_j.$$

By assuming $\rho(x) = \rho_j$ if $x \in I_j$, we identify $\rho(x)$ with a n-dimensional vector $\rho = [\rho_0, \rho_1, \ldots, \rho_{n-1}]^T \equiv \sum_j \rho_j e_j$ and represent $\hat{U}$ by a $n \times n$ matrix U_n defined as

$$(U_n)_{i,j} = \tfrac{1}{2}(\delta_{j,2i} + \delta_{j,2i+1} + \delta_{j+n,2i} + \delta_{j+n,2i+1}) \qquad i, j = 0, 1, \ldots, (n-1). \tag{1.3}$$

For example,

$$U_2 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad U_3 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \qquad U_4 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}. \tag{1.4}$$

We point out that U_n for even values of n is the transfer matrix of the well-known binary graph, which has been studied in many fields, e.g. combinatorics [6], quantum map [7] and quantum chaos [8–10].

Note that $U_n^T \neq U_n$ if $n > 3$ and hence U_n generally has complex eigenvalues. The equilibrium state $\rho_0 = \frac{1}{\sqrt{n}}[1, 1, \ldots, 1]^T$ is both the left and right eigenvector of U_n, i.e. $U_n \rho_0 = \rho_0$ and $U_n^T \rho_0 = \rho_0$. In this paper, we shall calculate all the eigenvalues and (left and right) eigenvectors of U_n and discuss their behaviour in the limit $n \to \infty$.

2. Eigenvalue

We start from the trace relation,

$$\sum_{j=1}^{n} \lambda_j^k = \mathrm{tr}\big(U_n^k\big) = \sum_{i=0}^{n-1} \big(U_n^k\big)_{ii} = \frac{1}{2^k} \sum_{i=0}^{n-1} m_i^{(k)}, \qquad k = 1, 2, \ldots, \tag{2.1}$$

where $m_i^{(k)}$'s are integers defined as follows. The action of F^k on I_i can be split into two consecutive steps. The first step is stretching, i.e. $I_i \to \big[2^k \frac{i}{n}, 2^k \frac{i+1}{n}\big] \equiv I_i^{(k)}$, and the second step is folding, i.e. rewinding $I_i^{(k)}$ to [0, 1]. The integer $m_i^{(k)}$ counts the number that $I_i^{(k)}$ coves I_i in the course of rewinding, which obviously equals to the number of fixed points of F^k in I_i. Because F^k has exactly 2^k fixed points and a fixed point at $\frac{i}{n}$ $(0 < i < n)$ contributes to both $m_{i-1}^{(k)}$ and $m_i^{(k)}$, we have

$$\mathrm{tr}\big(U_n^k\big) = 1 + \frac{1}{2^k} \sum_{i=1}^{n-1} \delta_{F^k(\frac{i}{n}), \frac{i}{n}}. \tag{2.2}$$

For a t-period PO contained in $\mathcal{P}_n \equiv \left\{\frac{i}{n}\middle| 0 < i < n\right\}$, its contribution to the above boundary correction of trace is

$$\frac{t}{2^k}\sum_{q=1}^{\infty}\delta_{k,qt} = \sum_{j=1}^{t}\left(\frac{1}{2}\,\mathrm{e}^{\mathrm{i}\frac{2\pi j}{t}}\right)^k. \tag{2.3}$$

Inserting equations (2.2) and (2.3) into equation (2.1), we have

$$\sum_{j=1}^{n}\lambda_j^k = 1 + \sum_{O_\alpha\in\mathcal{P}_n}\sum_{j=1}^{t_\alpha}\left(\frac{1}{2}\,\mathrm{e}^{\mathrm{i}\frac{2\pi j}{t_\alpha}}\right)^k, \tag{2.4}$$

where O_α represents a PO and t_α is its period. Therefore, except for $\lambda_0 = 1$, the non-zero eigenvalues of U_n are directly connected to PO in the finite set $\mathcal{P}_n$ of partition points. Specifically, a t-period PO produces t eigenvalues $\lambda = \frac{1}{2}\,\mathrm{e}^{\mathrm{i}\frac{2\pi j}{t}}$, $j = 1, 2, \ldots, t$. Based on this fact, we conclude that

(1) if $\lambda \neq 1$ is an eigenvalue of U_n, then $|\lambda| = \frac{1}{2}$ or 0,
(2) $U_{(2l+1)}$ has no zero eigenvalue, i.e. it is not degenerate,
(3) $U_{2^q(2l+1)}$ and $U_{(2l+1)}$ have identical set of non-zero eigenvalues; hence $U_{2^q(2l+1)}$ have $(2^q-1)(2l+1)$-fold zero eigenvalues.

3. Eigenvector

3.1. Non-degenerate case

Assume n is an odd number. In this case, each point in $\mathcal{P}_n$ belongs to a PO and U_n has $n-1$ eigenvalues on the circle $|\lambda| = \frac{1}{2}$. We first consider the right eigenvectors. The calculation can be greatly simplified by taking advantage of the relation $\hat{U}\hat{S} = \frac{1}{2}\hat{S}\hat{U}$, where $\hat{S}$ is a linear operator defined by

$$(\hat{S}f)(x) = \int_0^x f(x')\,\mathrm{d}x' - c, \tag{3.1}$$

where the constant $c = \int_0^1 (1-x')f(x')\,\mathrm{d}x'$, which ensures that $\int_0^1 (\hat{S}f)(x)\,\mathrm{d}x = 0$. Note that $\hat{U}f = \lambda f$ implies that $\hat{U}\hat{S}f = \frac{1}{2}\hat{S}\hat{U}f = \frac{\lambda}{2}\hat{S}f$, i.e. $\hat{S}$ transforms one eigenvector of $\hat{U}$ to another with halving eigenvalue. Suppose $O_\alpha \equiv \left\{\frac{\alpha_1}{n}, \frac{\alpha_2}{n}, \ldots, \frac{\alpha_{t_\alpha}}{n}\right\} \subset \mathcal{P}_n$ is a PO, then

$$\phi_{\alpha,k}(x) = \sum_{j=1}^{t_\alpha}\mathrm{e}^{-\mathrm{i}\frac{2\pi kj}{t_\alpha}}\,\delta\left(x - \frac{\alpha_j}{n}\right), \quad (0 \leqslant k < t_\alpha) \tag{3.2}$$

is an eigenvector of $\hat{U}$ with $\lambda = \mathrm{e}^{\mathrm{i}\frac{2\pi k}{t_\alpha}}$. Of course, $\phi_{\alpha,k}$ does not belong to the domain of U_n, however, $\hat{S}\phi_{\alpha,k}$ does since

$$\hat{S}\delta\left(x - \frac{\mathrm{i}}{n}\right) = \left(\frac{\mathrm{i}}{n} - 1\right)\sum_{j=0}^{i-1}e_j + \frac{\mathrm{i}}{n}\sum_{j=i}^{n-1}e_j \equiv \mathcal{H}_i. \tag{3.3}$$

The $n-1$ vectors $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_{n-1}$, are linearly independent and they expand the orthogonal complement of ρ_0. Therefore,

$$\mathcal{E}_{\alpha,k} = \hat{S}\phi_{\alpha,k} = \sum_{j=1}^{t_\alpha}\mathrm{e}^{-\mathrm{i}\frac{2\pi kj}{t_\alpha}}\mathcal{H}_{\alpha_j} \neq 0 \tag{3.4}$$

is a right eigenvector of U_n with $\lambda = \frac{1}{2}\,\mathrm{e}^{\mathrm{i}\frac{2\pi k}{t_\alpha}}$.

Now we consider the left eigenvectors. Let $\mathcal{D}_i = e_i - e_{i-1}, i = 1, 2, \ldots, (n-1)$. Obviously, $\langle \mathcal{D}_i, \rho_0 \rangle \equiv \mathcal{D}_i^T \rho_0 = 0$ and $\langle \mathcal{D}_i, \mathcal{H}_j \rangle = \delta_{i,j}$. Therefore, $\mathcal{D}_i$'s also expand the orthogonal complement of ρ_0. For $O_\alpha \subset \mathcal{P}_n$, define

$$\tilde{\mathcal{E}}_{\alpha,k} = \frac{1}{t_\alpha} \sum_{j=1}^{t_\alpha} \mathrm{e}^{\mathrm{i}\frac{2\pi kj}{t_\alpha}} \mathcal{D}_{\alpha_j}, \quad (0 \leqslant k < t_\alpha) \tag{3.5}$$

then we have $\langle \tilde{\mathcal{E}}_{\alpha,k}, \mathcal{E}_{\beta,j} \rangle = \delta_{\alpha,\beta}\delta_{k,j}$. This orthogonal relation implies

$$I = \rho_0 \rho_0^T + \sum_{O_\alpha \subset \mathcal{P}_n} \sum_{k=0}^{t_\alpha - 1} \mathcal{E}_{\alpha,k} \tilde{\mathcal{E}}_{\alpha,k}^T, \tag{3.6}$$

which immediately leads to the spectral decomposition of U_n,

$$U_n = U_n I = \rho_0 \rho_0^T + \sum_{O_\alpha \subset \mathcal{P}_n} \sum_{k=0}^{t_\alpha - 1} \frac{1}{2} \mathrm{e}^{\mathrm{i}\frac{2k\pi}{t_\alpha}} \mathcal{E}_{\alpha,k} \tilde{\mathcal{E}}_{\alpha,k}^T. \tag{3.7}$$

This decomposition shows $\tilde{\mathcal{E}}_{\alpha,k}$ is a left eigenvector of U_n, i.e. $U_n^T \tilde{\mathcal{E}}_{\alpha,k} = \frac{1}{2} \mathrm{e}^{\mathrm{i}\frac{2k\pi}{t_\alpha}} \tilde{\mathcal{E}}_{\alpha,k}$.

3.2. Degenerate case

Assume $n = 2^q(2l + 1)$. In this case, besides $2l + 1$ non-zero eigenvalues, which are identical with that of U_{2l+1}, U_n has $(2^q - 1)(2l + 1)$ zero eigenvalues. On the other hand, U_n has only $n/2$ linearly independent eigenvectors corresponding to $\lambda = 0$. Therefore, U_n can only be transformed to the direct sum of some Jordan blocks if $q > 1$. For simplicity, we consider here only $n = 2^q$. The general case can be viewed as, in a certain sense, the direct product of this most degenerate case ($n = 2^q$) and the non-degenerate case ($n = 2l + 1$) (see the appendix).

In the case of $n = 2^q$, it is convenient to employ the binary representation of integers $j = \sum_{l=0}^{q-1} s_l 2^l$ and write $e_j = e(s_{q-1} \ldots s_1 s_0) \equiv e(S)$. The action of U_n on $e(S)$ is

$$U_n e(S) = \frac{1}{2} \sum_{k=0}^{1} e(\mathcal{F}_k S)$$

where $\mathcal{F}_k S = s_{q-2} \cdots s_1 s_0 k, k = 0, 1$. Then we take the Hadamard transformation: for a binary sequence $\Sigma = \sigma_{q-1} \cdots \sigma_1 \sigma_0$, define

$$\phi(\Sigma) = 2^{-q/2} \sum_S (-1)^{\Sigma S} e(S) \tag{3.8}$$

where $\Sigma S = \sum_{i=0}^{q-1} \sigma_i s_i$. The 2^q Hadamard vectors build a orthonormal basis, i.e.

$$\langle \phi(\overline{\Sigma}), \phi(\Sigma) \rangle = \Pi_{i=0}^{q-1} \delta_{\overline{\sigma}_i \sigma_i} \equiv \delta_{\overline{\Sigma},\Sigma} \tag{3.9}$$

and hence

$$I = \sum_\Sigma \phi(\Sigma)\phi(\Sigma)^T. \tag{3.10}$$

U_n takes a very simple form, the Jordan canonical form in fact, in this basis,

$$U_n \phi(\Sigma) = \begin{cases} \phi(\mathcal{F}_0 \Sigma) & (\sigma_{q-1} = 0) \\ 0 & (\sigma_{q-1} = 1). \end{cases} \tag{3.11}$$

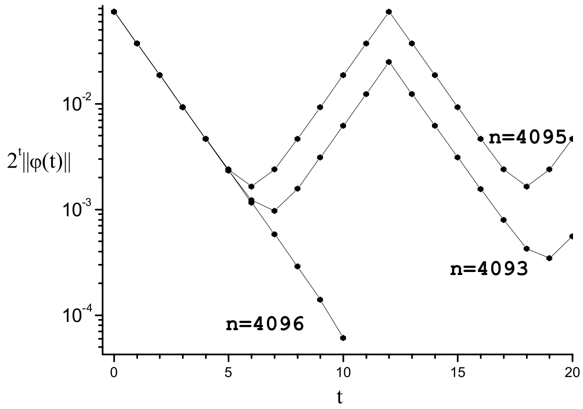


Figure 1. Evolution of $B_2(x)$ given by U_n. $B_2(x)$ is approximated by a n-dimensional vector $\varphi = \sum_{k=0}^{n-1}[B_2(\frac{2k+1}{2n}) + \frac{1}{12n^2}]e_k$ and $||\varphi|| \equiv \sqrt{\frac{1}{n}\langle\varphi, \varphi\rangle}$. Note that although the three matrices have distinct eigenvalues, they give the same short time decay $||\varphi(t)|| \sim 4^{-t}$.

Table 1. Jordan decomposition of U_{16}.

Basis of subspace	Jordan block
$\{\phi(0000)\}$	$J_1(1)$
$\{\phi(1000), \phi(0100), \phi(0010), \phi(0001)\}$	$J_4(0)$
$\{\phi(1001)\}$	$J_1(0)$
$\{\phi(1011)\}$	$J_1(0)$
$\{\phi(1101)\}$	$J_1(0)$
$\{\phi(1111)\}$	$J_1(0)$
$\{\phi(1010), \phi(0101)\}$	$J_2(0)$
$\{\phi(1110), \phi(0111)\}$	$J_2(0)$
$\{\phi(1100), \phi(0110), \phi(0011)\}$	$J_3(0)$

Consequently,

$$U_n = U_n I = U_n \sum_{\Sigma} \phi(\Sigma)\phi(\Sigma)^T = \sum_{\sigma_{q-2}=0}^{1} \cdots \sum_{\sigma_1=0}^{1}\sum_{\sigma_0=0}^{1} \phi(\sigma_{q-2}\cdots\sigma_1\sigma_0 0)\phi(0\sigma_{q-2}\cdots\sigma_1\sigma_0)^T. \tag{3.12}$$

This decomposition completely describes the spectral property of U_n, which can be summarized as

$$U_{2^q} \sim J_1(1) \oplus J_q(0) \oplus_{k=1}^{q-1} 2^{q-k-1} J_k(0), \tag{3.13}$$

where $J_k(\lambda)$ denotes the Jordan block, $J_k(\lambda)_{i,j} = \lambda\delta_{i,j} + \delta_{i,j-1}$, $1 \leqslant i, j \leqslant k$. Take $q = 4$ as an example, we have $U_{16} \sim J_1(1) \oplus J_4(0) \oplus 4J_1(0) \oplus 2J_2(0) \oplus J_3(0)$ (for detailed explanation, see table 1).

4. Large n limit

So far we have calculated the eigenvalues and eigenvectors of U_n for arbitrary n. The result deeply relies on the arithmetical properties, especially parity, of n and there is no simple limit

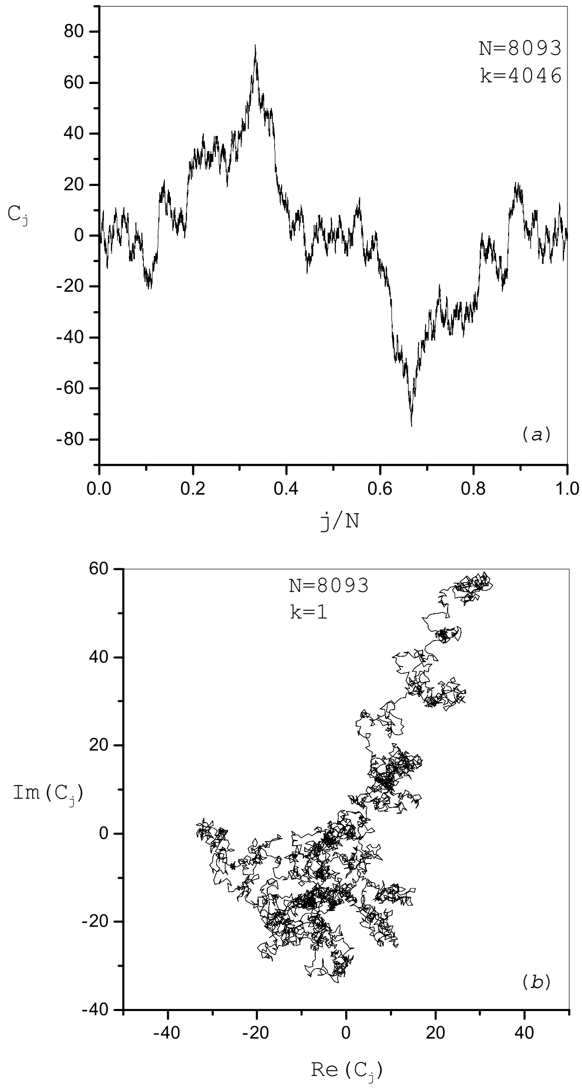


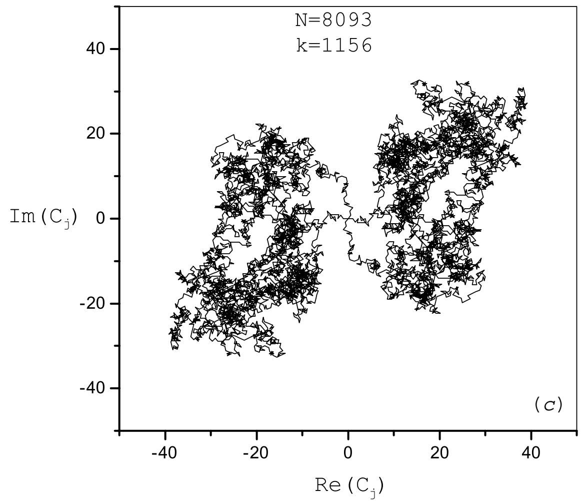
Figure 2. Some right eigenvectors of U_{8093}. $2\lambda = -1$, $e^{i\frac{2\pi}{8092}}$, $e^{i\frac{2\pi}{7}}$ and i for (*a*)–(*d*) respectively. Complex eigenvectors are represented by paths in the complex plane.

in respect of eigenvalues or eigenvectors when $n \to \infty$. Therefore, it is interesting to examine how U_n approaches $\hat{U}$ in the limit of large n.

We first consider the evolution given by U_n. Due to the exponential expanding of small cell, one can only expect that U_n can mimic $\hat{U}$ within a short time $\tau \sim \log_2 n$. This requirement does not conflict with the fact that U_n and $\hat{U}$ have different spectra. For example, if n is odd, the spectral decomposition equation (3.7) implies

$$\rho(t; n) = U_n^t \rho(0) = \langle \rho_0, \rho(0) \rangle \rho_0 + \frac{1}{2^t} \sum_{O_\alpha \subset \mathcal{P}_n} \sum_{k=0}^{t_\alpha - 1} e^{i\frac{2kt\pi}{t_\alpha}} \langle \tilde{\mathcal{E}}_{\alpha,k}, \rho(0) \rangle \mathcal{E}_{\alpha,k}. \quad (4.1)$$

This expression suggests that $\rho(t; n)$ approaches the equilibrium state as 2^{-t} when $t \to \infty$. However, for bounded t, different behaviour can be produced when $n \to \infty$. In this case,



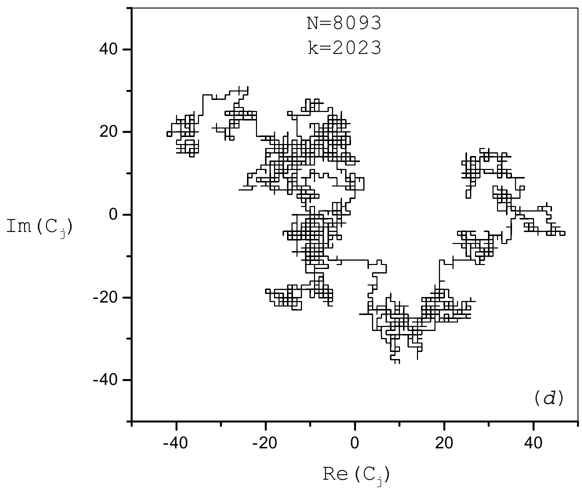


Figure 2. (Continued.)

all eigenvalues of U_n, except for $\lambda = 1$, are densely located on the circle $|\lambda| = 1/2$. This results an effective continuous spectrum, which can lead to complex short time dynamics, e.g. $\rho(t; n) \sim \lambda^t \rho(0)$ with $|\lambda| \neq 1/2$. Similarly, if $n = 2^q$, the $(n-1)$-fold zero eigenvalue does not mean that $\rho(t; n)$ will immediately approach ρ_0. In this case, the diversity of short time dynamics is caused by the Jordan blocks with increasing dimensionality (see figure 1).

Then we consider the eigenvectors. There is an eigenvector with $\lambda = 1/2$ defined as

$$\frac{1}{n} \sum_{O_\alpha \subset \mathcal{P}_n} \mathcal{E}_{\alpha,0} = \left[\frac{1-n}{2n}, \frac{3-n}{2n}, \ldots, \frac{n-3}{2n}, \frac{n-1}{2n}\right]^T, \tag{4.2}$$

which tends to $B_1(x)$ when $n \to \infty$. This is the only eigenvector which has a smooth limit. We should distinguish two cases in studying the limit behaviour of eigenvectors. In the first case we consider the eigenvectors associated with a fixed PO. When $n \to \infty$, according to equations (3.4) and (3.5), the right eigenvectors are uniquely determined by the PO and hence remain fixed while the left eigenvectors approach to singular functions, i.e. if $O_\alpha \subset \mathcal{P}_{n_0}$ and $n = (2l+1)n_0$,

$$n^2 \tilde{\mathcal{E}}_{\alpha,k} \sim \frac{1}{t_\alpha} \sum_{j=1}^{t_\alpha} \mathrm{e}^{\mathrm{i}\frac{2\pi kj}{t_\alpha}} \delta'\left(x - \frac{\alpha_j}{n_0}\right) \tag{4.3}$$

for $l \gg 1$. In the second case we simply let $n \to \infty$ and study all its eigenvectors. Then PO's with increasing period must be taken into account. Write $\mathcal{E}_{\alpha,k} = [c_0, c_1, \ldots, c_{n-1}]^T$, according to equation (3.4), the components are connected by $c_j = c_{j-1} + \Delta_j$ with $\Delta_j = \mathrm{e}^{-\mathrm{i}\frac{2\pi mk}{t_\alpha}}$ if $j = \alpha_m$ or 0 if $\frac{j}{n} \notin O_\alpha$. The chaotic nature of $F(x)$ implies that a long PO generally occupies the partition points in an apparent random order. Consequently, the components of $\mathcal{E}_{\alpha,k}$, as long as $k \neq 0$, can be locally viewed as a kind of Brownian motion in the complex plane. This is most evident in the situation when $\mathcal{P}_n$ consists of a single PO[1] (see figure 2). Again, the limit of thus eigenvectors cannot be smooth functions. This is consistent with the fact that $\hat{U}$ has only one smooth eigenfunction with $|\lambda| = 1/2$.

5. Conclusion

In this paper we have demonstrated with the Bernoulli map that the eigenvalues and eigenvectors of the discrete F–P operator can be extracted from the information of periodic orbits. In a more profound manner, this is believed to be true for general chaotic dynamical systems [2]. In addition, we have found that the naive expectation that a steady spectral density should be obtained when the partition of phase space is infinitely fine fails in this case. The reason can be ascribed to the spectral instability of non-Hermitian operators from the viewpoint of mathematics. It has been generally proved for expanding maps that only the isolated spectrum of the finite matrices outside a disc $|\lambda| = \lambda_c$ ($\lambda_c = 1/2$ in our case) is stable [11, 12]. Due to this fact, our study suggests it is impossible to give a definite description of long time behaviour in the framework of phase space discretization.

Acknowledgment

The author would like to thank Dr Jiao Wang for helpful advices to improve the manuscript.

Appendix. Jordan decomposition of $U_{2^q(2l+1)}$

In this appendix we consider the Jordan decomposition of $U_{2^q(2l+1)}$. For the sake of convenience, we adopt the outer product of vectors in the discussion. Let $A = (a_1, a_2, \ldots, a_r)^T$ and $B = (b_1, b_2, \ldots, b_s)^T$, $A \otimes B$ is a rs-dimensional vector defined as

$$A \otimes B = (a_1b_1, a_1b_2, \ldots, a_1b_s, a_2b_1, a_2b_2, \ldots, a_2b_s, \ldots, a_rb_1, a_rb_2, \ldots, a_rb_s)^T. \tag{A.1}$$

One can readily verify that

$$(A \otimes B) \otimes C = A \otimes (B \otimes C) \equiv A \otimes B \otimes C, \tag{A.2}$$

and

$$\langle A' \otimes B', A \otimes B\rangle = \langle A', A\rangle\langle B', B\rangle. \tag{A.3}$$

[1] The statement that $\mathcal{P}_n$ consists of a single PO is equivalent to that, in terms of number theory, 2 is a primitive root mod n. A necessary condition for this is that n is a prime number of the form of $8k + 3$ or $8k + 5$.

The construction of Hadamard vectors is an example of vector outer product. In fact, define

$$\phi(0) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \qquad \text{and} \qquad \phi(1) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} \tag{A.4}$$

we have

$$\phi(\sigma_{q-1}\sigma_{q-2}\cdots\sigma_0) = \phi(\sigma_{q-1}) \otimes \phi(\sigma_{q-2}) \otimes \cdots \otimes \phi(\sigma_0). \tag{A.5}$$

We can rewrite the action of U_n in terms of vector outer product. Let $V = [v_1, v_2, \ldots, v_k]^T$ be a k-dimensional vector, it can be easily verified that

$$U_{2k}\phi(\sigma) \otimes V = \delta_{0,\sigma} V \otimes \phi(0), \qquad U_{2k} V \otimes \phi(0) = (U_k V) \otimes \phi(0) \tag{A.6}$$

and similarly

$$U_{2k}^T V \otimes \phi(\sigma) = \delta_{0,\sigma}\phi(0) \otimes V, \qquad U_{2k}^T\phi(0) \otimes V = \phi(0) \otimes \left(U_k^T V\right). \tag{A.7}$$

Equation (A.6) imply $\operatorname{rank}(U_{2k}) = k$ and $\operatorname{rank}\left(U_{2k}^2\right) = \operatorname{rank}(U_k)$. Consequently, we have

$$\operatorname{rank}\left(U_{2^q(2l+1)}^m\right) = \begin{cases} 2^{q-m}(2l+1) & (0 < m \leqslant q) \\ 2l+1 & (q < m). \end{cases} \tag{A.8}$$

Note that the Jordan blocks of $U_{2^q(2l+1)}$ involve only $\lambda = 0$, according to theory of linear algebra, the above relation uniquely determines the form of its Jordan decomposition. In the following, we shall construct a basis that realizes this decomposition.

For a binary sequence $S = s_{q-1}\cdots s_1 s_0 \neq 0^q$, we cut it at the right of the leftmost 1 and split it into two segments $S = S_1 S_2$. Define

$$\psi(S, \mathcal{E}) = \phi(S_1) \otimes \mathcal{E} \otimes \phi(S_2) \equiv S_1 \mathcal{E} S_2,$$

where $\mathcal{E}$ is a $(2l+1)$-dimensional vector. For example, $\psi(1,\mathcal{E}) = 1\mathcal{E}, \psi(10,\mathcal{E}) = 1\mathcal{E}0, \psi(01,\mathcal{E}) = 01\mathcal{E}, \psi(11,\mathcal{E}) = 1\mathcal{E}1$ and so on. It can be shown that

$$\langle \psi(\overline{S}, \overline{\mathcal{E}}), \psi(S, \mathcal{E})\rangle = \delta_{\overline{S},S}\overline{\mathcal{E}}^T\mathcal{E}.$$

Therefore, if we take $\mathcal{E}$ from an orthonormal basis of R^{2l+1}, then the set of $\psi(S,\mathcal{E})$ builds an orthonormal basis of a $(2l+1)(2^q-1)$-dimensional subspace H_0, in which we have

$$U_{2^q(2l+1)}\psi(S,\mathcal{E}) = \begin{cases} \psi(\mathcal{F}_0 S, \mathcal{E}) & (s_{q-1} = 0) \\ 0 & (s_{q-1} = 1), \end{cases}$$

i.e. $U_{2^q(2l+1)}$ takes the Jordan canonical form. Furthermore, $\mathcal{E}\otimes\phi(0^q)$'s expand the complement (not orthogonal in general) of H_0, where $U_{2^q(2l+1)}$ is isomorphic to U_{2l+1}.

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